Impulse Response Inferences With Existence of Repeated Roots*

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Abstract

Vector Autoregression (VAR) and local projection (LP) are the two main methods of estimating and conducting inferences of the impulse response functions (IRFs) in macroeconomic studies, allowing researchers to choose between them based on the subjects of interest. This paper extends existing works on the comparison between AR inferences and LP inferences, by considering data generating processes with repeated roots. Consequently, the autoregressive estimation of impulse responses will converge to a special type of real-valued random variable, and the bootstrap Efron confidence interval of lag-augmented AR will always be conservative, even if the roots are away from the unit circle. This problem is more severe when the time series is highly persistent and at both intermediate and long horizons. The results are supported by Monte Carlo simulations with different values of roots in AR(2), AR(3) and VAR(1) models.

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1 Introduction

Estimating and conducting inference about impulse response functions is a central concern in quantitative economic studies that use time series data. Many methods have been developed to address key challenges and enhance the accuracy and robustness of these estimations and inferences. Among these methods, two stand out and are widely used in empirical research: the conventional *Vector Autoregression* (VAR) estimation and the *local projection* method introduced by Jordà (2005). However, there has been ongoing debate about which method is more suitable in different contexts. Recent literature has provided more insights into the comparative advantages of these two methods. Plagborg-Møller & Wolf (2021) demonstrated that VAR and local projection estimate the same impulse responses in the population, suggesting that future studies should focus more on their performance in finite samples. Montiel Olea & Plagborg-Møller (2021) addressed the issue of inference under different assumptions, showing that using additional lags in local projection yields robust inference results both at long horizons and with roots close to unity. Li et al. (2021) compared the two methods using a weighted loss function and found that local projection is generally less biased than VAR, while VAR typically produces lower variance at intermediate and long horizons, similar to the comparison between direct forecasting and iterated forecasting procedures (Schorfheide 2005, Marcellino et al. 2006). These studies provide researchers with guidelines on how to choose between the methods in empirical work, based on their data-generating processes (DGPs) or objective functions.

This paper extends the work of Montiel Olea & Plagborg-Møller (2021) and Inoue & Kilian (2020) by examining a specific case of data-generating processes (DGPs): AR and VAR models with repeated eigenvalues in the companion matrix[1]. In the original works, the lag-augmented AR method achieves correct nominal coverage at all horizons when the DGP is an AR(1)[2], although this comes with the drawback of relatively wide confidence intervals. This paper demonstrates that with repeated eigenvalues[3], the limiting distribution of the

[1]This result also applies when the DGP has two approximately equal roots. However, how inferences are related to the difference between the roots is beyond the scope of this paper.

[2]See Table 1 and Table 2 of Montiel Olea & Plagborg-Møller (2021).

[3]"Eigenvalues" and "roots" are used interchangeably throughout this paper. Since the roots refer to the companion matrix of the DGP, a stationary time series requires all of them to lie within the unit circle.

autoregressive estimation of impulse responses becomes atypical, and the bootstrap lag-augmented AR method's confidence intervals are always too conservative at intermediate and long horizons. In this paper, I provide theoretical proofs of this conservativeness for AR(2) and AR(3) models, and use Monte Carlo simulations with various parameter values across different horizons to support the results numerically. My simulations also show that the confidence intervals generated by lag-augmented (V)AR methods are very unstable in their sizes. With the existence of multiple roots, the size of the AR bootstrap CI can easily explode, even without unit roots or roots close to one.

The main contribution of this paper to the existing literature is twofold. From a theoretical perspective, it demonstrates that the bootstrap lag-augmented AR confidence interval consistently over-covers the true impulse response function (IRF) compared to the required nominal coverage probability at long horizons in finite samples, unlike the results for AR(1) presented in Montiel Olea & Plagborg-Møller (2021) and Inoue & Kilian (2020). From an empirical perspective, this paper highlights three key findings from Monte Carlo simulations: First, the lengths of the AR bootstrap confidence intervals (CIs) are highly dependent on the underlying DGPs and are very unstable compared to other methods. Therefore, researchers should be cautious when using it for time series inference and should ideally have prior knowledge of their data structure. Second, the lag-augmented local projection method consistently achieves the correct nominal level without the presence of a unit root, making it a robust estimation method for different DGPs, including cases with repeated roots. Third, Hall's percentile confidence interval for the lag-augmented AR method is also too conservative when there are repeated roots. This finding contrasts sharply with Inoue & Kilian (2020), which suggests that Hall's percentile CI for the lag-augmented AR method undercovers at all horizons except the shortest one[4].

This paper is structured as follows: Section 2 summarizes the two main methods for IRF estimation, including the uniform validity of the related inferences. Section 3 introduces the preliminaries of the theoretical results and states the main results with existence of repeated roots. Section 4 discusses the implementation of Monte Carlo simulations of AR(2) models.

[4]See Table 2 of Inoue & Kilian (2020).

Section 5 concludes. Proofs and Monte Carlo simulation results for AR(3) and VAR(1) models are included in the Appendix.

2 IRF Estimation: AR and LP

This section briefly summarizes the existing literature on IRF estimations in macroeconomics, and how the results of pointwise convergence evolves to the requirement of the uniform validity of the inferences. Throughout this paper, I focus on the simple AR(p) model and assume that the lag length p is known by the econometricians.

It is well-known that conventional OLS estimation and inference of the time series model face severe size distortions when the time series is highly persistent, i.e. there is a unit root or root close to the unit circle. The seminal work of Mikusheva (2007) summarized a few proposed method to solve the inference problem with such roots, while raising the question of whether these methods provide uniformly valid confidence intervals. Specifically, pointwise convergence only satisfies that there exists a large enough sample size to achieve the required nominal level for each value in the parameter space, however the required sample size can be extremely large for some values in the parameter space and lead to poor coverage performances in the finite sample. On the contrary, uniform convergence has a stronger requirement that:

$$\liminf_{T\to\infty} \inf_{\rho\in\Theta} P_\rho\{\rho \in \widehat{C}\} \geq 1-\alpha$$

Hence, the coverage probability is achieved asymptotically regardless of the value in the parameter space for some large enough sample size. Mikusheva (2007) further showed that pointwise asymptotics is not sufficient for uniform asymptotic validity, by proving that the subsampling interval proposed by Romano & Wolf (2001) satisfies the former but not the latter.

Recent literature on IRF estimations focused on the correct coverage probability and uniform validity of the inferences at the same time. The leading methods, which are discussed as follows, are lag-augmented autoregression discussed in Inoue & Kilian (2020) and lag-augmented local projection proposed by Montiel Olea & Plagborg-Møller (2021).

2.1 Lag-Augmented Autoregression

I follow the notations in Inoue & Kilian (2020) to introduce the framework of lag-augmented AR inference. Consider an AR model of order p:

$$y_t = \rho_1 y_{t-1} + \cdots + \rho_p y_{t-p} + u_t$$

for $t = 1, \ldots, T$ and u_t is a time series of white noise. The textbook AR method allows us to estimate the AR slope coefficients with regression, and then use the plug-in estimations to recover the impulse response functions based on iterated formulae.

Instead of fitting this AR(p) model, the *lag-augmented* AR method (Toda & Yamamoto 1995)(Dolado & Lütkepohl 1996) fits one more lag of outcome variable y_{t-p-1}:

$$y_t = \rho_1 y_{t-1} + \cdots + \rho_p y_{t-p} + \rho_{p+1} y_{t-p-1} + u_t$$

but only uses the first p estimated coefficients $\hat{\rho}_1, \ldots, \hat{\rho}_p$ to recover the impulse response functions for different horizons of interest h and conduct impulse response inferences. The use of additional lag helps to rule out the singularity of the asymptotic variance of a continuously differentiable function $f(\theta)$ of parameter θ from the model (Kilian & Lütkepohl 2017) and achieve uniformity even with non-stationary time series.

They require the following assumptions[5] to derive the uniform validity of the inferences:

1. The roots of the polynomial $|\rho(z) = 0|$ have at most one outside the unit circle, whereas all others must be inside the unit circle in modulus.

2. u_t has bounded fourth moment.

Based on the assumptions, Inoue & Kilian (2020) showed that the bootstrap approximations have uniform validity, by proving the following results:

$$\lim_{T\to\infty} \sup_{\theta\in\Theta} \sup_{x\in\mathcal{R}} \left| P^*((Df(\hat{\theta}_T^*)\hat{\Sigma}_T^* Df(\hat{\theta}_T^*)')^{-1/2}(f(\hat{\theta}_T^*) - f(\hat{\theta}_T)) \leq x) - \Phi(x) \right| = 0$$

[5] See Assumption A (Inoue & Kilian 2020)

$$\lim_{T\to\infty} \sup_{\theta\in\Theta} \sup_{x\in\mathcal{R}} \left| P^*((f(\hat{\theta}_T^*) - f(\hat{\theta}_T))(Df(\hat{\theta}_T^*)\hat{\Sigma}_T^* Df(\hat{\theta}_T^*)')^{-1}(f(\hat{\theta}_T^*) - f(\hat{\theta}_T)) \leq x) - F_{\chi^2_{d_\Psi}}(x) \right| = 0$$

almost surely conditional on the data $Y^T = (y_1, \ldots, y_T)$. In the above result, superscript asterisk denotes the bootstrap version of the estimators, $\Phi(\cdot)$ denotes the cumulative density function of a standard normal distribution and $F_{\chi^2_{d_\Phi}}(\cdot)$ represents the cumulative density function of the chi-squared distribution with degrees of freedom d_Ψ.

2.2 Lag-Augmented Local Projection

On the other hand, an increasing number of studies start to adopt the method of local projection (LP) introduced by Jordà (2005), as it is easy to understand and implement in empirical works. Whereas the AR method is using recursive forecasting, local projection is an analogy to direct forecast procedure (Ramey 2016). However, whether local projection inference has some advantages compared to VAR inferences has long been under debate. Although Jordà (2005) suggested that local projection both simplifies the inference and is more robust to model misspecifications, these conclusions were rejected by Kilian & Lütkepohl (2017). Recently, the work of Breitung et al. (2019), Plagborg-Møller & Wolf (2021) and Montiel Olea & Plagborg-Møller (2021) allowed people to have a deeper understanding of the role of local projection inferences in macroeconomics.

Using the same DGP as that of the AR method, the researcher is interested in the IRF at h periods ahead, denoted by $\beta(\rho, h)$, where $\rho = (\rho_1, \ldots, \rho_p)$. The original DGP can be written in the following way:

$$y_{t+h} = \beta(\rho, h)y_t + \sum_{l=1}^{p-1} \delta_l(\rho, h)y_{t-l} + \xi_t(\rho, h)$$

where

$$\xi_t(\rho, h) = \sum_{l=1}^{h} \beta(\rho, h-l)u_{t+l}$$

The conventional local projection method basically regresses y_{t+h} on p lags of y_t, and apply the heteroskedasticity and autocorrelation consistent/robust (HAC/HAR) standard errors for inference ((Jordà 2005), (Ramey 2016)). On the other hand, the lag-augmented local projection method regress y_{t+h} on y_t, while controlling for additional p periods $(y_{t-1}, \ldots, y_{t-p})$. Hence, it is similar to the process of lag-augmentation in AR method as it regresses on $p+1$ lags of outcome variables. The only additional assumption is that we require the residual term $\{u_t\}$ to be strictly stationary and $\mathbb{E}[u_t|\{u_s\}_{s\neq t}] = 0$ almost surely. Let $x_t \equiv (y_t, \ldots, y_{t-p})'$, we get:

$$\begin{pmatrix} \hat{\beta}(h) \\ \hat{\gamma}(h) \end{pmatrix} \equiv \left(\sum_{t=1}^{T-h} x_t x_t' \right)^{-1} \sum_{t=1}^{T-h} x_t y_{t+h}$$

Another interesting feature of lag-augmented LP is that it suffices to use the heteroskedasticity-robust Eicker-Huber-White standard error defined as

$$\hat{s}(h) \equiv \frac{(\sum_{t=1}^{T-h} \hat{\xi}_t(h)^2 \hat{u}_t(h)^2)^{1/2}}{\sum_{t=1}^{T-h} \hat{u}_t(h)^2}$$

where

$$\hat{\xi}_t(h) \equiv y_{t+h} - \hat{\beta}(h) y_t - \hat{\gamma}(h)' \tilde{x}_t, \quad \tilde{x}_t \equiv (y_{t-1}, \ldots, y_{t-p})'$$
$$\hat{u}_t(h) \equiv y_t - \hat{\rho}(h)' \tilde{x}_t, \quad \hat{\rho}(h) \equiv \left(\sum_{t=1}^{T-h} \tilde{x}_t \tilde{x}_t' \right)^{-1} \left(\sum_{t=1}^{T-h} \tilde{x}_t y_t \right)$$

The confidence interval with nominal size $100(1-\alpha)\%$ is given by:

$$\hat{C}(h, \alpha) \equiv [\hat{\beta}(h) - z_{1-\alpha/2} \hat{s}(h) \ , \ \hat{\beta}(h) + z_{1-\alpha/2} \hat{s}(h)]$$

Montiel Olea & Plagborg-Møller (2021) also derived the uniform validity of lag-augmented LP inferences by proving that:

$$\inf_{\lambda \in [-1,1]} \inf_{1 \leq h \leq \bar{h}_T} P_\rho(\beta(\rho, h) \in \widehat{C}(h, \alpha)) \to 1 - \alpha \text{ as } T \to \infty$$

The uniformity of the inference requires that either the roots are not at the unit circle, or the horizon of interest h_T satisfies $h_T/T \to 0$.

3 Main Results: AR(2) Model

This section first lists the preliminaries for the main result in the AR(2) model with repeated eigenvalues, then states the main results that: 1. The autoregressive estimation of impulse responses converges to a special distribution. 2. Lag-augmented AR bootstrap confidence interval is always conservative. Consider the following AR(2) model:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t$$

with companion matrix

$$A \equiv \begin{pmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues of the companion matrix are given as

$$\lambda_1 = \frac{\rho_1}{2} + \sqrt{\frac{\rho_1^2}{4} + \rho_2}, \ \lambda_2 = \frac{\rho_1}{2} - \sqrt{\frac{\rho_1^2}{4} + \rho_2}$$

Hence,

$$\lambda_1 = \lambda_2 \ \Leftrightarrow \ \frac{\rho_1^2}{4} + \rho_2 = 0$$

If the eigenvalues are different, A has the following eigenvalue decomposition:

$$A = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

If $\lambda_1 = \lambda_2 = \lambda$, the companion matrix has the following Jordan decomposition

$$A = S \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} S^{-1}$$

where

$$S = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$$

The true impulse response function at horizon h is give by

$$\text{IRF}(h) = (1,0)A^h(1,0)' = \begin{cases} \frac{\lambda_1^{h+1}-\lambda_2^{h+1}}{\lambda_1-\lambda_2} & \text{if } \lambda_1 \neq \lambda_2 \\ \lambda^h(h+1) & \text{if } \lambda_1 = \lambda_2 = \lambda \end{cases}$$

Throughout the case of AR(2) model, I keep the following assumptions:

Assumption 1. (Repeated Roots and Asymptotic Normality)

1. $\lambda_1 = \lambda_2 = \lambda = \frac{\rho_1}{2}$
2. The plug-in IRF is based on AR estimators $(\hat{\rho}_1, \hat{\rho}_2)$ that are asymptotically bivariate normal, i.e.

$$T^{1/2}\begin{pmatrix}\hat{\rho}_1 - \rho_1 \\ \hat{\rho}_2 - \rho_2\end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}_2(0, W)$$

The following proposition states the result that when there are repeated roots, the autoregressive estimation of impulse response function converges in distribution to a special real-valued random variable that is different from that in the usual cases.

Proposition 1.

Let the horizon of interest $h_T = hT^{1/4}$, then

$$\frac{\widehat{IRF}(h_T)}{IRF(h_T)} \xrightarrow{d} \frac{\lambda}{h\sqrt{Z}}\sinh(\frac{h}{\lambda}\sqrt{Z})$$

where $Z \equiv (\lambda, 1)(Z_1, Z_2)'$ and the limiting distribution is a real-valued random variable.

Proof of Proposition 1. See Appendix.

I then consider the inference on the impulse response function at given horizons using bootstrap IRF estimators. Let $\rho^* = (\rho_1^*, \rho_2^*)'$ be the consistent bootstrap version of the estimators for $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2)'$. Define

$$\beta(\theta_T^*, \phi_T^*; Y^T) \equiv \sup_{f \in BL(1,\mathbb{R}^p)} \left|\mathbb{E}[f(\theta_T^*)|Y^T] - \mathbb{E}[f(\phi_T^*)|Y^T]\right|$$

as the *bounded Lipschitz distance* between the distributions induced by θ_T^* and ϕ_T^* conditional on the data $Y^T = (y_1, \ldots, y_T)$. The random vectors θ_T^* and ϕ_T^* are said to converge in *bounded Lipschitz distance in probability* (conditional on Y^T) if

$$\beta(\theta_T^*, \phi_T^*; Y^T) \xrightarrow{p} 0 \text{ as } T \to \infty$$

I derive the convergence property of bootstrap impulse response functions, and propose an approximation of its $1-\alpha$ quantile in the following proposition. Using the approximation, the proposition shows that the lag-augmented AR(2) bootstrap quantile over-covers the true IRF at horizon h_T:

Proposition 2.

(i) Let $(Z_1^*, Z_2^*)' \sim \mathcal{N}(0, W)$ be a bivariate normal vector independent of the data. Suppose that

$$\beta(T^{1/2}(\rho^* - \hat{\rho}), (Z_1^*, Z_2^*)) \xrightarrow{p} 0$$

then

$$\beta\left(\frac{IRF^*(h_T)}{(\hat{\rho}_1/2)^{h_T} h_T}, \frac{\sinh(f^*(Z^* + \tilde{Z}))}{f^*(Z^* + \tilde{Z})}\right) \xrightarrow{p} 0$$

where

$$Z^* = (\hat{\rho}_1/2, 1)(Z_1^*, Z_2^*)'$$

$$\tilde{Z} = T^{1/2}\left(\frac{\hat{\rho}_1^2}{2} + 4\hat{\rho}_2\right)$$

and

$$f^*(x) = \frac{h\sqrt{x}}{\hat{\rho}_1/2}$$

Moreover, we can use the quantile of the following term to approximate the quantile of the $1-\alpha$ bootstrap quantile of the IRFs:

$$(\hat{\rho}_1/2)^{h_T} h_T \frac{\sinh(f^*(Z^* + \tilde{Z}))}{f^*(Z^* + \tilde{Z})}$$

and we can approximate it by

$$(\hat{\rho}_1/2)^{h_T} h_T \hat{c}_{1-\alpha} = (\hat{\rho}_1/2)^{h_T} h_T \cdot g\left(\sqrt{z_{1-\alpha} + \sqrt{T}(\hat{\rho}_1/2 + 4\hat{\rho}_2)}\frac{h}{\hat{\rho}_1/2}\right)$$

where $g(x) \equiv \frac{\sinh(x)}{x}$.

(ii) The $1-\alpha$ *two-sided Efron bootstrap CI can be approximated by*

$$\widehat{C} \equiv \left[\hat{c}_{\alpha/2}(\hat{\rho}_1/2)^{h_T} h_T,\ \hat{c}_{1-\alpha/2}(\hat{\rho}_1/2)^{h_T} h_T\right]$$

whereas $h_T = hT^{1/4}$ *and*

$$\hat{c}_\alpha = g\left(\sqrt{z_\alpha + \sqrt{T}(\hat{\rho}_1/2 + 4\hat{\rho}_2)}\frac{h}{\hat{\rho}_1/2}\right)$$

whereas $g(x) \equiv \frac{\sinh(x)}{x}$.
The coverage probability of the Efron bootstrap CI is

$$P\left(IRF(h) \in \widehat{C}\right) > 1 - \alpha$$

Proof of Proposition 2. See Appendix.

Remark 1.

We can extend the proof of AR(2) to AR(3). In the Appendix, we only consider the case when the AR(3) model has two repeated roots. Note that in this case, the plug-in estimators for the repeated roots will be conjugate pairs and have exactly the opposite asymptotic distributions with convergence rate of $T^{1/4}$. Only the distinct root will have an asymptotic distribution close to normal with a common convergence rate of $T^{1/2}$. We also need to decentralize the IRF estimator in AR(3) model to get the same limiting distribution as in AR(2) cases. For the details of differences in proofs, please see Appendix.

Proposition 1 and 2 state the main results of this paper based on AR(2) model with repeated eigenvalues. The Efron bootstrap CI should cover the true IRF more frequently than the required nominal level. This is different from results in previous implementations that the bootstrap CI always achieves the correct coverage probability, across all DGPs. However, since both local projection and lag-augmented local projection are based on directly regressing the outcome variable at horizon h periods ahead onto the current observation and potential controls, they do not involve using plug-in estimators and hence should not show such over-coverage problems in inferences. Moreover, lag-augmented local projection is suggested to have the correct size across all data generating processes of AR and VAR, unless the researcher is interested in long horizon impulse responses with unit root.

4 Monte Carlo Simulation Results

Following the propositions in the last section, this section shows the Monte Carlo simulation results for AR(2) model. The sample size is $T = 240$, with values of the repeated root $\lambda = \{0, 0.5, 0.95, 1\}$ and horizons $h = \{1, 6, 12, 36, 60\}$. Each Monte Carlo simulation has 5000 repetitions and 2000 draws for each bootstrap implementation. The targeted nominal confidence level is 90% across all cases.

I consider four candidates in the simulation: (1) conventional non-augmented AR estimator with a delta method CI, (2) lag-augmented AR estimator with bias-adjusted Efron bootstrap CI (Efron 1992), (3) non-augmented LP estimator with HAC standard errors, Hall's percentile-t CI (Hall 1992), and (4) lag-augmented LP estimator with AR bootstrap, Hall's percentile-t CI suggested in Montiel Olea & Plagborg-Møller (2021), Section 5. Table 1 reports the coverage probabilities and median lengths of the confidence intervals in each setup.

It is not surprising that we see some of the simulation results close to the heuristic AR(1) implementation in Montiel Olea & Plagborg-Møller (2021):

1. The conventional non-augmented AR asymptotics does not achieve the correct coverage probabilities in many cases. When the root is zero, the asymptotic normality fails because the Jacobian matrix with respect to the impulse response function is singular (Benkwitz et al. 2000). When the asymptotic normality holds as $\lambda = 0.5$, the con-

	Coverage				Median Length			
h	AR	ARLA	LP_b	$LPLA_b$	AR	ARLA	LP_b	$LPLA_b$
				$\lambda = 0$				
1	0.900	0.899	0.918	0.907	0.212	0.212	0.234	0.219
6	1.000	0.894	0.906	0.893	0.001	0.004	0.233	0.219
12	1.000	0.892	0.908	0.904	0.000	0.000	0.231	0.222
36	1.000	0.894	0.902	0.904	0.000	0.000	0.242	0.235
60	1.000	0.893	0.898	0.899	0.000	0.000	0.260	0.251
				$\lambda = 0.5$				
1	0.897	0.899	0.923	0.910	0.205	0.212	0.229	0.219
6	0.877	0.887	0.899	0.893	0.278	0.477	0.417	0.382
12	0.847	0.947	0.904	0.910	0.039	0.125	0.401	0.378
36	0.948	0.934	0.905	0.905	0.000	0.001	0.427	0.402
60	0.945	0.928	0.899	0.910	0.000	0.000	0.465	0.435
				$\lambda = 0.95$				
1	0.878	0.905	0.903	0.912	0.095	0.214	0.102	0.220
6	0.873	0.885	0.906	0.908	1.405	3.578	1.707	1.846
12	0.865	0.892	0.898	0.895	3.558	29.001	4.911	4.280
36	0.853	0.944	0.827	0.777	9.629	> 1000	14.846	11.727
60	0.809	0.938	0.781	0.812	11.817	> 1000	15.248	10.383
				$\lambda = 1$				
1	0.545	0.828	0.824	0.895	0.052	0.214	0.055	0.222
6	0.515	0.666	0.851	0.878	0.999	3.535	1.178	2.312
12	0.486	0.839	0.862	0.866	3.321	34.894	4.363	6.608
36	0.399	0.964	0.781	0.812	18.226	> 1000	30.835	37.325
60	0.336	0.961	0.710	0.746	36.258	> 1000	69.466	78.831

Table 1: MONTE CARLO RESULTS: AR(2), T = 240

ventional AR is conservative as shown in the last section. When the root is closer to one and asymptotic normality assumption fails, we see the well-known size distortions. However, non-augmented AR method is still favorable when the researchers are dealing with short horizon estimations with roots that are far away from unity, because it is more efficient than the other methods (Montiel Olea & Plagborg-Møller 2021) in these cases.

2. Following the proof in the last section, the Efron bootstrap CI with lag-augmented AR method is always conservative at both intermediate and long horizons across all DGPs except when the time series is of no persistence, but it behaves relatively well when the researchers are interested in IRFs at short horizons. The problem of over-coverage

will be alleviated if the length of the time series increases and $h/T \to 0$. However, we may still recommend to use lag-augmented AR method at intermediate and long horizons in practice, because the variance is substantially smaller than those of the LP methods (Plagborg-Møller & Wolf 2021). The second concern with the lag-augmented AR method is that the length of the confidence interval is extremely unstable and highly depends on the underlying DGP: in the case of $\rho \leq 0.5$, the median lengths of the CI shrink to zero as the true IRF shrinks to zero rather fast. In local-to-unity root or unit root cases, the median lengths of AR bootstrap CI explodes (Montiel Olea & Plagborg-Møller 2021), compared to all other three candidates.

3. The non-augmented local projection method estimates the impulse response function by regressing y_{t+h} directly on y_t and y_{t-1}, without controlling for additional lag and use HAR/HAC standard error corrections. The sizes of the Hall's percentile-t confidence interval are close to the nominal level, when the roots are away from the unit circle. It also has less size distortion compared to conventional AR methods. Compared to lag-augmented local projection, it sometimes has a slightly higher efficiency, because it has regressors with larger variances compared to lag-augmented local projection.

4. Finally, lag-augmented local projection achieves the correct coverage probability at most cases without unit root and long horizon inference, and the median lengths of the confidence intervals are not exploding in extreme cases. It is relatively robust to different setups. However, due to the under-coverage problem with unit root or local-to-unity roots in highly persistent data, researchers may still prefer lag-augmented AR methods to lag-augmented local projection, if they favor more conservative results instead of higher probability of type I error. Note that the requirement for using EHW standard error for the inference is slightly stronger than the usual assumption that the innovations to be a martingale difference sequence.

Remark 2.

1. Shown in Table 2, the Hall's percentile confidence interval of lag-augmented AR method also suffers from an over-coverage problem, and they are more conservative compared

to their Efron bootstrap counterparts. Such result is different from the conclusions in Inoue & Kilian (2020), that the Hall percentile CI is not recommended because of its poor coverage probability at all horizons but the shortest one. The proof for this case is similar to the proof for Efron bootstrap confidence interval, as Hall's percentile CI is constructed based on $\hat{c}_{\alpha/2}$ and $\hat{c}_{1-\alpha/2}$ as well.

2. The coverage probabilities of LPLA for $\lambda = 0.95$ and $h = 36, 60$ are not very ideal, because we have a combination of local-to-unity roots and long horizons compared to the sample size. When the sample size increases, LPLA will achieve the correct nominal level with roots of 0.95, and the conservativeness of ARLA bootstrap CI will also be alleviated[6].

3. The Monte Carlo simulation results for AR(3) and VAR(1) models are similar to those for AR(2) model. In AR(3) model, I consider both cases of a largest distinct root and a smallest distinct root. The simulation results show that the over-coverage problem is more significant when the largest root is distinct and the two smaller roots are identical. The simulation results are included in Appendix B.

4. In the VAR(1) simulation, I use the same DGP as in Kilian & Kim (2011):

$$y_t = \begin{pmatrix} B_{11} & 0 \\ 0.5 & 0.5 \end{pmatrix} y_{t-1} + e_t, \quad e_t \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}\right)$$

and set B_{11} as 0.5 to create repeated eigenvalues for the companion matrix in VAR(1) case. Since the value of the root is not local-to-unity, the other three methods all gets the same correct nominal size of confidence intervals, with reasonably small median lengths. The result is somehow opposite to the impression from Montiel Olea & Plagborg-Møller (2021), that the lag-augmented AR bootstrap method is uniformly valid across all DGPs, no matter for short, intermediate or long horizon inferences.

5. Similar to the implementation results of AR(1) in existing literature, the median lengths of the Efron bootstrap CI vary significantly with respect to the underlying

[6] In the simulation with $T = 1200$ and $\lambda = 0.95$, LPLA has the correct coverage at all horizons, and ARLA is conservative only at $h = 60$.

DGPs. Note that the length of the Efron bootstrap CI is given by

$$(\hat{c}_{1-\alpha/2} - \hat{c}_{\alpha/2})(\hat{\rho}_1/2)^{h_T} h_T$$

and $\hat{\rho}_1 = \hat{\lambda}_1 + \hat{\lambda}_2$ in AR(2) model. Hence, when the repeated root is closer to one, and when the horizon of interest h_T increases, the size of the $1-\alpha$ confidence interval increases as well. This problem is more significant when the underlying DGP is of higher order, because the second part of the CI corresponds to the plug-in estimation of IRF, and it is easy for the true IRF to be large with higher order, as long as the largest eigenvalue is large. This result is also proved in Montiel Olea & Plagborg-Møller (2021), Appendix B.2.2, as they argued that the bootstrap CI converges in distribution to a non-degenerate random variable.

	Coverage				Median Length			
ρ	0	0.5	0.95	1	0	0.5	0.95	1
$h = 1$	0.898	0.892	0.897	0.883	0.212	0.212	0.214	0.214
$h = 6$	0.551	0.953	0.992	0.998	0.004	0.477	3.578	3.535
$h = 12$	0.596	0.818	0.995	0.996	0.000	0.125	29.001	34.894
$h = 36$	0.569	0.906	0.991	0.910	0.000	0.000	4.015e+03	5.379e+03
$h = 60$	0.561	0.910	0.998	0.984	0.000	0.000	6.596e+05	1.028e+06

Table 2: MONTE CARLO RESULTS: AR(2) HALL'S PERCENTILE CI, T = 240

5 Conclusion

The recent series of papers comparing existing estimation methods for impulse responses have given researchers great insights in the usage of them in finite sample, especially when the researchers are dealing with highly persistent time series data or having interest in inferences at long horizons. This paper contributes to existing literature by considering a special setup in data generating processes: the existence of repeated eigenvalues of the companion matrix. I show that the bootstrap AR confidence interval is always conservative at both intermediate and long horizons compared to the sample size, because the repeated roots lead to a strange limiting distribution of the estimated impulse responses. Such result is different from those in existing works, changing the impression that bootstrap AR inference always achieves the required nominal level. Moreover, the simulation results show that the lengths of the bootstrap confidence interval highly depend on the underlying DGP, and it is easier for the confidence interval to explode when the model is VAR and of higher order.

In general, the result suggests that: with non-persistent time series data and short horizon inference, the textbook AR inference outperforms other competitors with higher efficiency. Lag-augmented AR is more favorable when there is a high chance of unit root, however the researchers are interested in inferences at short horizons or care less about possible conservativeness and exploded confidence intervals. Lag-augmented local projection will be a safe choice, as long as the horizon of interest is not very long while there is a unit root or root that is local-to-unity. However, this method requires that innovations satisfy the mean independence assumption, which is slightly stronger.

Several extensions could be pursued based on the simple showcases of this paper. First, it will be interesting to see whether such property remains with repeated roots in AR(∞) and VAR models with higher order, although it might be tedious to consider many more possibilities of root distributions. A more general theoretical results built on the current work may be ideal. Second, as someone may argue, under-coverage are of higher interest and concerns to researchers compared to over-coverage problems. Hence, it remains an open question that whether the distribution of roots will cause under-coverage problems as well. Finally, it will be interesting to discuss the relationship between this conservativeness and the feature of iterated forecasting of AR method, and compare it to the non-conservativeness of direct forecasting of local projection methods.

Appendix A PROOFS OF RESULTS

A.1 Proof of Proposition 1.

I first prove the following lemma, which derives the the asymptotic distribution of the AR estimators of the roots $\hat{\lambda}_1$ and $\hat{\lambda}_2$:

Lemma 1.

Given Assumption 1, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ have the following distribution with convergence rate of $T^{1/4}$:

$$T^{1/4}\begin{pmatrix}\hat{\lambda}_1 - \lambda \\ \hat{\lambda}_2 - \lambda\end{pmatrix} \xrightarrow{d} \begin{pmatrix}\sqrt{Z} \\ -\sqrt{Z}\end{pmatrix}$$

where $Z \equiv (\lambda, 1)(Z_1, Z_2)'$.

Proof of Lemma 1.

$$\begin{aligned}
&T^{1/4}(\hat{\lambda}_1 - \lambda) \\
=&T^{1/4}\left(\frac{\hat{\rho}_1}{2} + \sqrt{\frac{\hat{\rho}_1^2}{4} + \hat{\rho}_2} - \frac{\rho_1}{2}\right) \\
=&\frac{T^{1/4}}{T^{1/2}}T^{1/2}\frac{1}{2}(\hat{\rho}_1 - \rho_1) + \sqrt{T^{1/2}\left(\frac{\hat{\rho}_1{}^2}{4} + \hat{\rho}_2 - \frac{\rho_1^2}{4} - \rho_2\right)} \\
\xrightarrow{d}&\sqrt{(\lambda, 1)(Z_1, Z_2)'}
\end{aligned}$$

as $T \to \infty$, the first term vanishes. The last equality follows from applying delta method and using the fact that $\frac{\rho_1}{2} = \lambda$. The proof of the limiting distribution of $\hat{\lambda}_2$ is analogous to the above process.

Using the result in Lemma 1, we can derive the limiting distribution of the AR estimation for impulse response function at horizon $h_T = hT^{1/4}$:

$$
\begin{aligned}
\frac{\widehat{IRF}(h_T)}{IRF(h_T)} &= \frac{\hat{\lambda}_1^{h_T+1} - \hat{\lambda}_2^{h_T+1}}{\hat{\lambda}_1 - \hat{\lambda}_2} \frac{1}{\lambda^{h_T}(h_T+1)} \\
&= \left\{ \left(\frac{\hat{\lambda}_1}{\lambda_1}\right)^{h_T+1} - \left(\frac{\hat{\lambda}_2}{\lambda_2}\right)^{h_T+1} \right\} \frac{\lambda}{(\hat{\lambda}_1 - \hat{\lambda}_2)(h_T+1)} \\
&= \{\exp\left((h_T+1)(\ln(\hat{\lambda_1}) - \ln(\lambda_1))\right) - \exp\left((h_T+1)(\ln(\hat{\lambda_2}) - \ln(\lambda_2))\right)\} \\
&\quad \times \frac{\lambda}{T^{1/4}(\hat{\lambda}_1 - \hat{\lambda}_2)(T^{-1/4}(h_T+1))} \\
&\xrightarrow{d} \left\{ \exp\frac{h}{\lambda}\sqrt{Z} - \exp -\frac{h}{\lambda}\sqrt{Z} \right\} \frac{\lambda}{2h\sqrt{Z}} \\
&= \frac{\lambda}{h\sqrt{Z}} \sinh(\frac{h}{\lambda}\sqrt{Z})
\end{aligned}
$$

where the last equality follows from $\sinh(x) \equiv (e^x - e^{-x})/2$.

To show that the limiting distribution is a real-valued random variable, note that in the case of $Z < 0$, the limiting distribution is:

$$
\begin{aligned}
& \left\{ \exp i\frac{h}{\lambda}\sqrt{Z} - \exp i\left(-\frac{h}{\lambda}\sqrt{Z}\right) \right\} \frac{\lambda}{2hi\sqrt{Z}} \\
= & \frac{2i\sin\left(\frac{h}{\lambda}\sqrt{-Z}\cdot\lambda\right)}{2hi\sqrt{Z}} \\
= & \sin\left(\frac{\lambda}{h}\sqrt{-Z}\right) \Big/ \frac{h}{\lambda}\sqrt{-Z}
\end{aligned}
$$

which is a real-valued random variable. The case when $Z > 0$ is trivial. □

A.2 Proof of Proposition 2.

(i) First, I prove that

$$
\beta\left(\frac{IRF^*(h_T)}{(\hat{\rho}_1/2)^{h_T} h_T}, \frac{\sinh(f^*(Z^* + \tilde{Z}))}{f^*(Z^* + \tilde{Z})} \right) \xrightarrow{p} 0
$$

The proof is analogous to that of Proposition 1, with the following adjustment:

$$
\begin{aligned}
T^{1/4}(\hat{\lambda}_1 - \hat{\rho}_1/2) &= \frac{T^{1/4}}{T^{1/2}} T^{1/2}(\rho_1^* - \hat{\rho}_2) + \sqrt{T^{1/2}\left(\frac{\rho_1^{*2}}{4} + \rho_2^*\right)} \\
&= \frac{T^{1/4}}{T^{1/2}} T^{1/2}(\rho_1^* - \hat{\rho}_2) \\
&+ \sqrt{T^{1/2}[(\rho_1^{*2}/4 + \rho_2^*) - (\hat{\rho}_1^2/4 + \hat{\rho}_2)] + T^{1/2}(\hat{\rho}_1^2/4 + \hat{\rho}_2)} \\
&\xrightarrow{d} \sqrt{Z^* + \tilde{Z}}
\end{aligned}
$$

replacing λ by $\frac{\hat{\rho}_1}{2}$ and $\sqrt{Z}$ by $\sqrt{Z^* + \tilde{Z}}$ yields the desired result. Hence, we can approximate the $1 - \alpha$ quantile of bootstrap estimator $IRF^*(h_T)$ by the quantile of

$$
(\hat{\rho}_1/2)^{h_T} h_T \frac{\sinh(f^*(Z^* + \tilde{Z}))}{f^*(Z^* + \tilde{Z})}
$$

(ii) Note that the function

$$
g(x) \equiv \sinh(x)/x
$$

is monotonically increasing in $x > 0$. Therefore, the function g has an inverse denoted by $g^{-1}(\cdot)$. If $Z^* + \tilde{Z} < 0$, then

$$
\frac{\sinh(f^*(Z^* + \tilde{Z}))}{f^*(Z^* + \tilde{Z})} < 1
$$

For any constant $c > 1$, we have:

$$
\begin{aligned}
& P(g(f^*(Z^* + \tilde{Z})) \leq c) \\
= & P(g(f^*(Z^* + \tilde{Z})) \leq c \ \& \ Z^* + \tilde{Z} > 0) + P(g(f^*(Z^* + \tilde{Z})) \leq c \ \& \ Z^* + \tilde{Z} \leq 0) \\
= & P(f^*(Z^* + \tilde{Z}) \leq g^{-1}(c) \ \& \ Z^* + \tilde{Z} > 0) + P(Z^* \leq -\tilde{Z}) \\
= & P(h\sqrt{Z^* + \tilde{Z}}/(\hat{\rho}_1/2) \geq g^{-1}(c) \ \& \ Z^* + \tilde{Z} > 0) + P(Z^* \leq -\tilde{Z}) \\
= & P\left(0 \leq Z^* + \tilde{Z} \leq g^{-1}(c)^2 \left(\frac{\hat{\rho}_1/2}{h}\right)^2\right) + P(Z^* \leq -\tilde{Z}) \\
= & P\left(Z^* \leq g^{-1}(c)^2 \left(\frac{\hat{\rho}_1/2}{h}\right)^2 - T^{1/2}(\hat{\rho}_1^2/2 + 4\hat{\rho}_2)\right)
\end{aligned}
$$

Hence, the $1 - \alpha$ quantile will satisfy

$$
z_{1-\alpha} = g^{-1}(c_{1-\alpha})^2 \left(\frac{\hat{\rho}_1/2}{h}\right)^2 - T^{1/2}(\hat{\rho}_1^2/2 + 4\hat{\rho}_2)
$$

implying

$$
\hat{c}_{1-\alpha} = g\left(\sqrt{z_{1-\alpha} + \sqrt{T}(\hat{\rho}_1/2 + 4\hat{\rho}_2)}\frac{h}{\hat{\rho}_1/2}\right)
$$

Given the above result, the coverage probability of the one-sided Efron bootstrap CI is given by

$$
\begin{aligned}
P\left(IRF(h_T) \leq \hat{c}_{1-\alpha}(\hat{\rho}_1/2)^{h_T} h_T\right) & \approx P\left((\rho_1/2)^{h_T} \leq \hat{c}_{1-\alpha}(\hat{\rho}_1/2)^{h_T}\right) \\
& = P(\exp(-h_T(\ln(\hat{\rho}_1) - \ln(\rho_1))) \leq \hat{c}_{1-\alpha}) \\
& \xrightarrow{d} P\left(1 \leq g\left(\sqrt{z_{1-\alpha} + Z}\frac{h}{\rho_1/2}\right) \ \& \ Z > -z_{1-\alpha}\right) \\
& + P(1 \leq Q \ \& \ Z < -z_{1-\alpha})
\end{aligned}
$$

whereas Z is defined as in Lemma 1 and Q is approximated quantile when $Z < -z_{1-\alpha}$. Since

$$
g > 1 \Leftrightarrow z_{1-\alpha} + Z > 0 \Leftrightarrow Z > -z_{1-\alpha}
$$

we have

$$P(Z > -z_{1-\alpha}) > 1 - \alpha$$

Hence, the Efron one-sided confidence interval under bootstrap lag-augmented AR reveals an over-coverage problem. The proof for a two-sided confidence interval follows similarly. □

A.3 Sketch of Proof for AR(3)

Suppose without loss of generality that $\lambda_2 = \lambda_3 = \lambda$ in the AR(3) case. We derive the results in AR(3) case to find the approximation of the $1 - \alpha$ bootstrap quantile of the IRFs.

Analogous to the AR(2) case, assume that

$$T^{1/2} \begin{pmatrix} \hat{\rho}_1 - \rho_1 \\ \hat{\rho}_2 - \rho_2 \\ \hat{\rho}_3 - \rho_3 \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} \sim \mathcal{N}_3(0, W)$$

then

$$T^{1/4} \begin{pmatrix} \hat{\lambda}_2 - \lambda_2 \\ \hat{\lambda}_3 - \lambda_3 \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \sqrt{Z} \\ -\sqrt{Z} \end{pmatrix}$$

where $Z \equiv (\lambda + \lambda_1, 1)(Z_1, Z_2)'$. Since the limiting distribution of $\hat{\lambda}_2$ and $\hat{\lambda}_3$ sum up to zero, applying the fact that $\rho_1 = \lambda_1 + \lambda_2 + \lambda_3$, we have

$$T^{1/2}(\hat{\lambda}_1 - \lambda_1) \xrightarrow{d} N(0, \sigma_1)$$

Define

$$\mu(h) \equiv \frac{\lambda_1^{h+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}$$

and denote $\hat{\mu}(h)$ its sample analogue. The term $\mu(h)$ provides the large h approximation of the IRFs when $\lambda_1 \neq \lambda_2 \neq \lambda_3$.

Proposition 3. *Suppose the assumptions above are satisfied and let* $h_T = hT^{1/4}$*, then*

$$\frac{\widehat{IRF}(h_T) - \hat{\mu}(h_T)}{IRF(h_T) - \mu(h_T)} \xrightarrow{d} \frac{\lambda}{h\sqrt{Z}} \sinh(\frac{h}{\lambda}\sqrt{Z})$$

Proof of proposition 3.

Numerator:

$$(\widehat{IRF}(h_T) - \hat{\mu}(h_T)) = -\frac{\hat{\lambda}_2^{h_T+2}}{(\hat{\lambda}_1 - \hat{\lambda}_2)(\hat{\lambda}_2 - \hat{\lambda}_3)} + \frac{\hat{\lambda}_3^{h_T+2}}{(\hat{\lambda}_1 - \hat{\lambda}_3)(\hat{\lambda}_2 - \hat{\lambda}_3)}$$

Denominator:

$$\begin{aligned}(IRF(h_T) - \mu(h_T)) &= -\frac{\lambda^{h_T+2}}{(\lambda_1 - \lambda)^2} - \frac{\lambda^{h_T+1}(h_T+2)}{\lambda_1 - \lambda} \\ &= \lambda^{h_T+2}(h_T+2)\left(-\frac{1}{(h_T+2)(\lambda_1 - \lambda)^2} - \frac{1}{\lambda(\lambda_1 - \lambda)}\right)\end{aligned}$$

The ratio is divided into two parts:

$$\begin{aligned}&\left(-\frac{1}{(h_T+2)(\lambda_1 - \lambda)^2} - \frac{1}{\lambda(\lambda_1 - \lambda)}\right)^{-1} \\ &= \left(\frac{-\lambda - (h_T+2)(\lambda_1 - \lambda)}{\lambda(\lambda_1 - \lambda)^2(h_T+2)}\right)^{-1} \\ &\xrightarrow{p} -\frac{\lambda(1-\lambda)^2(h_T+2)}{\lambda + (h_T+2)(1-\lambda)}\end{aligned}$$

and

$$\left(-\left(\frac{\hat{\lambda}_2}{\lambda}\right)^{h_T+2} \frac{1}{(\hat{\lambda}_1 - \hat{\lambda}_2)(h_T+2)(\hat{\lambda}_2 - \hat{\lambda}_3)} + \left(\frac{\hat{\lambda}_3}{\lambda}\right)^{h_T+2} \frac{1}{(\hat{\lambda}_1 - \hat{\lambda}_3)(h_T+2)(\hat{\lambda}_2 - \hat{\lambda}_3)}\right)$$

$$\xrightarrow{d} \left(-\exp\left(\frac{h}{\lambda}\sqrt{Z}\right) + \exp\left(-\frac{h}{\lambda}\sqrt{Z}\right)\right) \frac{1}{(1-\lambda)(h_T+2)(\hat{\lambda}_2 - \hat{\lambda}_3)}$$

Combine the two parts, we have:

$$
\begin{aligned}
&-\frac{\lambda(1-\lambda)^2(h_T+2)}{\lambda+(h_T+2)(1-\lambda)}\cdot\frac{1}{(1-\lambda)(h_T+2)(\hat{\lambda}_2-\hat{\lambda}_3)}\left(-\exp\left(\frac{h}{\lambda}\sqrt{Z}\right)+\exp\left(-\frac{h}{\lambda}\sqrt{Z}\right)\right)\\
=&-\frac{\lambda(1-\lambda)}{[\lambda+(h_T+2)(1-\lambda)](\hat{\lambda}_2-\hat{\lambda}_3)}\left(-\exp\left(\frac{h}{\lambda}\sqrt{Z}\right)+\exp\left(-\frac{h}{\lambda}\sqrt{Z}\right)\right)\\
=&-\frac{1-\lambda}{[1+\frac{(h_T+2)(1-\lambda)}{\lambda}](\hat{\lambda}_2-\hat{\lambda}_3)}\left(-\exp\left(\frac{h}{\lambda}\sqrt{Z}\right)+\exp\left(-\frac{h}{\lambda}\sqrt{Z}\right)\right)\\
\xrightarrow{d}&-\frac{\lambda}{(h_T+2)(\hat{\lambda}_2-\hat{\lambda}_3)}\left(-\exp\left(\frac{h}{\lambda}\sqrt{Z}\right)+\exp\left(-\frac{h}{\lambda}\sqrt{Z}\right)\right)\\
\xrightarrow{d}&\frac{\lambda}{h\sqrt{Z}}\left(\frac{\exp\left(\frac{h}{\lambda}\sqrt{Z}\right)-\exp\left(-\frac{h}{\lambda}\sqrt{Z}\right)}{2}\right)\\
=&\frac{\lambda}{h\sqrt{Z}}\sinh\left(\frac{h}{\lambda}\sqrt{Z}\right)
\end{aligned}
$$

where the last equality follows from $\sinh(x)\equiv(e^x-e^{-x})/2$ and that $\sinh(-x)=-\sinh(x)$. This result is parallel to that of the AR(2) case. Finally, from the above results and analogous to the proof of Proposition 1, it can be shown that

$$
\beta\left(\frac{IRF^*(h_T)-\mu^*(h_T)}{\widehat{IRF}(h_T)-\hat{\mu}(h_T)},\frac{\sinh(f^*(Z^*+\tilde{Z}))}{f^*(Z^*+\tilde{Z})}\right)\xrightarrow{p}0
$$

where

$$
\begin{aligned}
Z^*&=(\hat{\rho}_1/2+\hat{\lambda}_1,1)(Z_1^*,Z_2^*)'\\
\tilde{Z}&=T^{1/2}\left(\frac{(\rho_1-\hat{\lambda}_1)^2}{4}+\hat{\rho}_1+\hat{\rho}_2-1\right)
\end{aligned}
$$

and

$$
f^*(x)=\frac{h\sqrt{x}}{(\hat{\rho}_1-\hat{\lambda}_1)/2}
$$

Appendix B MONTE CARLO RESULTS

B.1 Monte Carlo Simulation of AR(3)

Table 3 summarizes the Monte Carlo simulation results for AR(3) model with two repeated roots. The top panel shows the simulation with distinct root $\lambda_1 = 0.8$ and repeated roots $\lambda_2 = \lambda_3 = 0.5$, and the bottom panel shows the result with distinct root $\lambda_1 = 0.5$ and repeated roots $\lambda_2 = \lambda_3 = 0.8$.

The results are not very different from those in the AR(2) cases. As this simulation does not use very extreme value of roots, both LP and LPLA behaves well. The bootstrap ARLA confidence interval has a more significant over-coverage problem with a large distinct root, as shown in the top panel. When the number of lag increases, it becomes easier for the confidence interval to explode when using lag-augmented AR method for IRF inferences.

Finally, it may be interesting to explore why the conventional AR method under-covers in the top panel but over-covers in the bottom panel. This may shed light on the understanding of the relationship between parameter values and coverage probabilities in other methods.

	Coverage					Median Length			
h	AR	ARLA	LP$_b$	LPLA$_b$		AR	ARLA	LP$_b$	LPLA$_b$
					$\lambda_1 = 0.8$				
1	0.890	0.879	0.909	0.902		0.206	0.213	0.228	0.221
6	0.846	0.873	0.893	0.893		1.470	2.636	1.758	1.662
12	0.806	0.895	0.868	0.863		2.170	16.844	1.283	1.266
36	0.786	0.964	0.902	0.909		4.162	71.287	1.267	1.232
60	0.756	0.962	0.904	0.910		5.990	3.800e+04	1.374	1.341
					$\lambda_1 = 0.5$				
1	0.890	0.889	0.907	0.900		0.202	0.214	0.222	0.221
6	0.884	0.894	0.902	0.906		1.607	2.726	1.969	1.752
12	0.886	0.891	0.876	0.882		2.342	19.490	2.671	2.631
36	0.955	0.925	0.899	0.904		0.702	709.750	2.610	2.536
60	0.967	0.913	0.904	0.910		0.052	3.425e+04	2.825	2.740

Table 3: MONTE CARLO RESULTS: AR(3), T = 240

B.2 Monte Carlo Simulation of VAR(1)

Table 4 shows the simulation results of the VAR(1) model described in Kilian & Kim (2011), with a repeated root of 0.5 and intercept normalized to zero. This model serves as a benchmark case in a variety of existing empirical literature. We observe the same over-coverage problem in VAR(1) model, even when the eigenvalues are not extreme.

	Coverage				Median Length			
h	VAR	VARLA_b	LP_b	LPLA_b	VAR	VARLA_b	LP_b	LPLA_b
1	0.895	0.892	0.904	0.903	0.209	0.224	0.230	0.232
6	0.885	0.900	0.904	0.902	0.136	0.045	0.360	0.364
12	0.836	0.920	0.908	0.910	0.018	0.045	0.360	0.364
36	0.876	0.930	0.903	0.902	0.000	0.000	0.382	0.385
60	0.873	0.926	0.899	0.905	0.000	0.000	0.408	0.413

Table 4: MONTE CARLO RESULTS: VAR(1), T = 240

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